APPENDIX I

AI DERIVATION OF THE HUGONIOT EQUATIONS

The Hugoniot equations relate the pressure P, internal energy per unit mass E, and density ρ in front of a shock wave (P_0, E_0, ρ_0) to the values of the same variables (P, E, ρ) after the shock wave has passed. The density is sometimes expressed in terms of specific volume $V = 1/\rho$, and $V_0 = 1/\rho_0$. The initial pressure, internal energy, and density are assumed to be known. In addition to the final pressure, energy, and density, the shock velocity U and particle velocity u_p behind the shock are unknown (the reference frame is usually chosen so that the unshocked material is at rest). The Hugoniot equations use the conservation of mass, momentum, and energy across the shock front to reduce the number of unknowns from five to two. The equation of state then provides a relation between the pressure, internal energy, and density to completely determine the conditions behind the shock wave.

Many derivations of the Hugoniot equations have been published. I have found the one below to be one of the easiest for students to follow.

AI.1 Mass conservation

Figure AI.1 illustrates a block of material through which a shock wave is passing. This is a "free-body" diagram in the sense that all forces acting on the block are explicitly shown. The cross-sectional area of the block A is constant as the shock moves through it. The pressures P and P_0 on the block's sides are not shown in the figure to avoid clutter: they are completely balanced and so play no direct role in driving the shock, so they are ignored in this derivation. Only the pressures acting on the ends of the block, in the direction of the shock wave's motion, are significant.

The figure shows the block at two different times, t and t'. At the earliest time t, the length of the unshocked region is l_u and the length of the shocked region is l_s. Later, at time t', the shock wave has progressed a distance $U(t' - t)$ farther to the right and the shocked end of the block, moving at the particle velocity u_p, has progressed $u_p(t' - t)$ farther to the right. The unshocked end of the block, assumed to be at rest, has not moved. The new lengths of the unshocked region l_u' and shocked region l_s' are thus given by

$$l_u' = l_u - U(t' - t) \qquad \text{(AI.1.1a)}$$

$$l_s' = l_s + U(t' - t) - u_p(t' - t) \qquad \text{(AI.1.1b)}$$

The mass contained in the unshocked portion of the block at time t is its volume l_uA times its density ρ_0, $\rho_0 l_u A$. The mass in the shocked portion is likewise $\rho l_s A$. Mass conservation just means that the masses at times t and t' must be equal. That is,

$$\rho l_s A + \rho_0 l_u A = \rho l_s' A + \rho_0 l_u' A \qquad \text{(AI.1.2)}$$

Canceling through the common factor A, substituting Equations AI.1.1a and AI.1.1b for l_s' and l_u' in Equation AI.1.2, then canceling the terms ρl_s and $\rho_0 l_u$ on both the left and right hand sides,

$$0 = \rho\,(U - u_p)\,(t' - t) - \rho_0\,U\,(t' - t) \qquad \text{(AI.1.3)}$$

Finally, canceling the common factor $(t' - t)$ and rearranging, we obtain the first Hugoniot equation (3.4.1) of the text

$$\rho\,(U - u_p) = \rho_0\,U \qquad \text{(AI.1.4)}$$

AI.2. Momentum conservation

Pressure P on the shocked end of the block in Figure AI.1 is larger than the pressure P_0 on the unshocked end, so a net force $F = (P - P_0)\,A$ acts toward the right, accelerating material in that direction. The momentum of material in the block at time t, $\rho l_s u_p A$, is thus not equal to the momentum $\rho l_s' u_p A$ at time t'. The difference is equal to the momentum imparted by the applied force F over the time interval $t' - t$, $F(t' - t)$. The net momentum balance is thus

$$\rho l_s'\,u_p A - \rho l_s\,u_p\,A = (P - P_0)A(t' - t) \qquad \text{(AI.2.1)}$$

Canceling through the area A and substituting for l_s' using Equation AI.1.1b, noting that the term $\rho l_s u_p$ is subtracted from itself on the left side of the equation, and further canceling the common factor $(t' - t)$ from the remaining terms we obtain

$$\rho(U - u_p)u_p = (P - P_0) \qquad \text{(AI.2.2)}$$

Now use Equation AI.1.4 to replace $\rho(U - u_p)$ by $\rho_0 U$ and rearrange to obtain the second Hugoniot

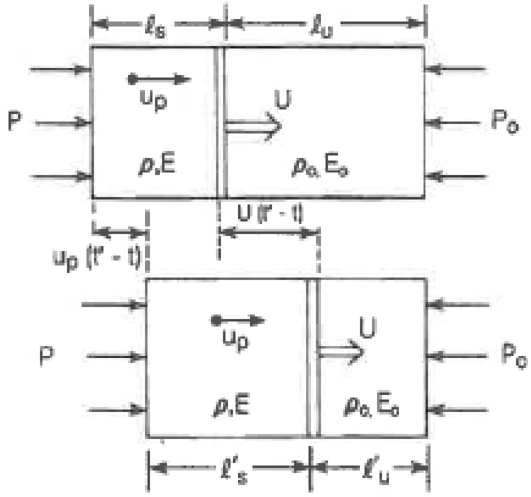

Fig. A1.1 Free body diagrams of a shock wave passing through a mass of material at times t (above) and t' (below).

equation (3.4.2) in the text,

$$P - P_0 = \rho_0 U u_p \qquad \text{(AI.2.3)}$$

AI.3. Energy conservation

Like momentum, the total energy in the block at time t is not equal to that at time t' because the applied forces do work on the system. This work is equal to the force times the distance through which it acts. Since the displacement of the unshocked end of the block is zero, the total energy gained between t and t' is thus $PAu_p(t' - t)$, equal to the force PA on the shocked end of the block times the distance $u_p(t' - t)$ through which it acts.

The total energy $E_{tot}(t)$ in the block at time t is the sum of the internal energies in the shocked and unshocked portions and the kinetic energy in the shocked portion:

$$E_{tot}(t) = \rho_0 l_u E_0 A + \rho l_s E A + 1/2\, \rho l_s u_p^2 A \qquad \text{(AI.3.1a)}$$

Similarly, at time t' the total energy is

$$E_{tot}(t') = \rho_0 l_u' E_0 A + \rho l_s' E A + ½\, \rho l_s' u_p^2 A \qquad \text{(A.3.1b)}$$

Energy conservation thus requires

$$E_{tot}(t') - E_{tot}(t) = PAu_p(t' - t) \qquad \text{(A1.3.2)}$$

Substituting Equations AI.3.1a and AI.3.1b into Equation AI.3.2, cancel A through as before, substitute Equation AI.1.1a for l'_u and Equation AI.1.1b for l'_s, and simplify. The common factor $(t' - t)$ may then be canceled to obtain

$$-\rho_0 E_0 U + \rho E(U - u_p) + ½\, \rho u_p^2 (U - u_p) = P u_p \qquad \text{(AI.3.3)}$$

Now replace $\rho(U - u_p)$ by $\rho_0 U$ using the first Hugoniot equation (AI.1.4) to obtain

$$\rho_0 U(E - E_0) + ½\, \rho_0 u_p^2 U = P u_p \qquad \text{((AI.3.4)}$$

We then proceed using two auxiliary equations that can be readily derived from the first two Hugoniot equations (AI.1.4) and (AI.2.3) by eliminating either U or u_p, respectively, between the two equations:

$$u_p = \sqrt{(P - P_0)(V - V_0)} \qquad \text{(AI.3.5a)}$$

and

$$U = 1/\rho_0 \sqrt{(P - P_0)/(V - V_0)} \qquad \text{(AI.3.5b)}$$

where $V = 1/\rho$ and $V_0 = 1/\rho_0$ are the specific volumes of the shocked and unshocked material, respectively. Substituting Equation AI.3.5a for u_p and AI.3.5b for U in Equation AI.3.4, canceling the common factor $\sqrt{(P - P_0)}$, and rearranging,

$$E - E_0 = ½\,(P + P_0)(V_0 - V) \qquad \text{(AI.3.6)}$$

which is the third, and final, Hugoniot equation (3.4.3) of the text.

$E\left(\frac{Nm}{Kg}\right)$